Topological invariants of line arrangements

B. Guerville-Ballé

Université de Pau et des Pays de l'Adour
Universidad de Zaragoza

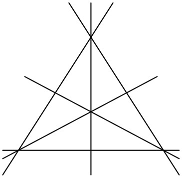
January 16, 2014

Advisors : E. Artal Bartolo, V. Florens, J. Vallès.

Arrangements & topology

Definition

A *complex line arrangement* $\mathcal{A}$ is a set of lines $\{L_0, \cdots, L_n\}$ of $\mathbb{CP}^2$.



Ceva's arrangement (1678)

Definition

The *topological type* of an arrangement $\mathcal{A}$ is the homeomorphism type of the pair $(\mathbb{CP}^2, \bigcup L_i)$.

Combinatorics

Definition

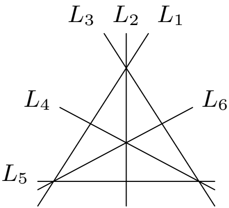
The *combinatorics* of an arrangement $\mathcal{A}$ is a triplet $(\mathcal{A}, \mathcal{Q}, \in)$ composed of :

- the set of lines : $\mathcal{A}$,
- the set of intersection points : $\mathcal{Q}$,
- the incidence relations between them : $\in$.

Example

The combinatorics of Ceva's arrangement is :

$$[\ [1,2,3],[1,4],[1,5,6],[2,4,6],[2,5],[3,4,5],[3,6]\ ]$$



Motivations

What is the influence of the combinatorics on the topology ?

Motivations

What is the influence of the combinatorics on the topology ?

Topological invariants :

- Complement : $E_{\mathcal{A}} = \mathbb{CP}^2 \setminus \mathcal{A}$.
- Fundamental group : $\pi_1(E_{\mathcal{A}})$.
- Invariant of the fundamental group :
  - character $\xi \in \mathrm{Hom}(\pi_1(E_{\mathcal{A}}); \mathbb{C}^*)$
  - Alexander's module
  - characteristic varieties

Motivations

What is the influence of the combinatorics on the topology ?

- (1980) P. Orlik and L. Solomon : *"Combinatorics and topology of complements of hyperplanes"*, (Inventiones Mathematicæ)

- (2001) E. Hironaka : *"Boundary manifold of line arrangements"*, (Mathematische Annalen)

- (2001) A. Libgober : "*Characteristic varieties of algebraic curves*", (NATO Science Series II : Mathematics, Physics and Chemistry)

- (2013) E. Artal : "*Topology of arrangement and position of singularities*", (Annales de la faculté des sciences de Toulouse)

- (2013) A. Dimca, D. Ibadula and D. Anca : "*Pencil type line arrangements of low degree : classification and monodromy*"

Motivations

What is the influence of the combinatorics on the topology ?

• (1998) G. Rybnikov : "*On the fundamental group of the complement of a complex hyperplane arrangement*",

• (2005) E. Artal, J. Carmona, J.I. Cogolludo and M. Marco : "*Topology and combinatorics of real line arrangements*", (Compositio Mathematica)

Motivations

What is the influence of the combinatorics on the topology ?

• (1998) G. Rybnikov : "*On the fundamental group of the complement of a complex hyperplane arrangement*",

• (2005) E. Artal, J. Carmona, J.I. Cogolludo and M. Marco : "*Topology and combinatorics of real line arrangements*", (Compositio Mathematica)

Definition

A *Zariski pair* is a pair of arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with :

- the same combinatorics,
- different topological type.

Results obtained

Two explicit descriptions of the inclusion map i_*

Results obtained

Two explicit descriptions of the inclusion map i_*

↳ Generalization of E. Hironaka's results on $i_* : B_{\mathcal{A}} \to E_{\mathcal{A}}$.

↳ Generalization of R. Randell's Theorem on presentation of $\pi_1(E_{\mathcal{A}})$.

Results obtained

Two explicit descriptions of the inclusion map i_*

↳ Generalization of E. Hironaka's results on $i_* : B_{\mathcal{A}} \to E_{\mathcal{A}}$.

↳ Generalization of R. Randell's Theorem on presentation of $\pi_1(E_{\mathcal{A}})$.

New topological invariant of arrangements

Results obtained

Two explicit descriptions of the inclusion map i_*

↳ Generalization of E. Hironaka's results on $i_* : B_{\mathcal{A}} \to E_{\mathcal{A}}$.

↳ Generalization of R. Randell's Theorem on presentation of $\pi_1(E_{\mathcal{A}})$.

New topological invariant of arrangements

↳ New examples of Zariski pairs.

↳ Algorithm to compute the quasi-projective depth of a character.

Strategy & plane

What is the influence of the combinatorics on the topology ?

Strategy & plane

What is the influence of the combinatorics on the topology ?

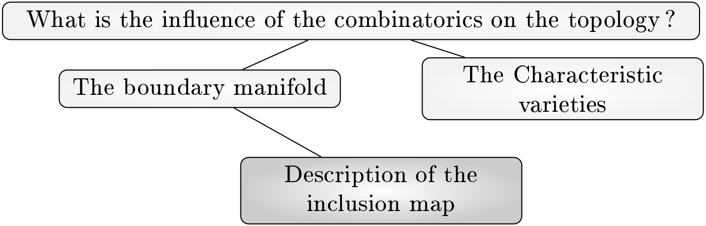
The boundary manifold

Strategy & plane

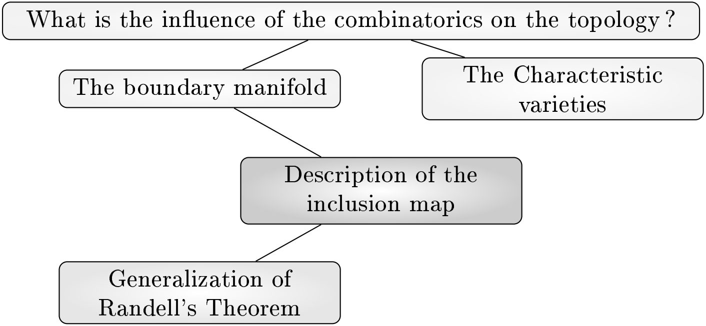
What is the influence of the combinatorics on the topology ?

- The boundary manifold
- The Characteristic varieties

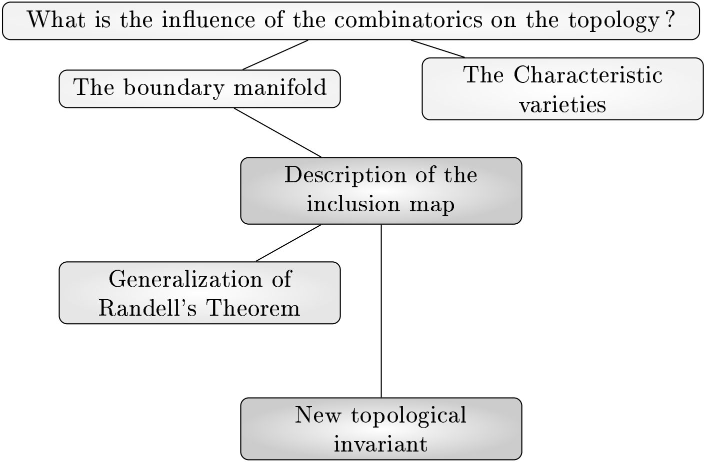
Strategy & plane



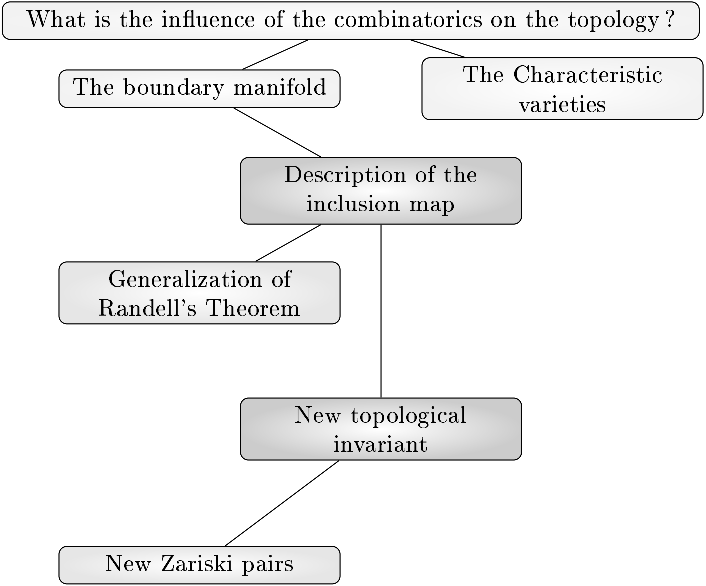
Strategy & plane



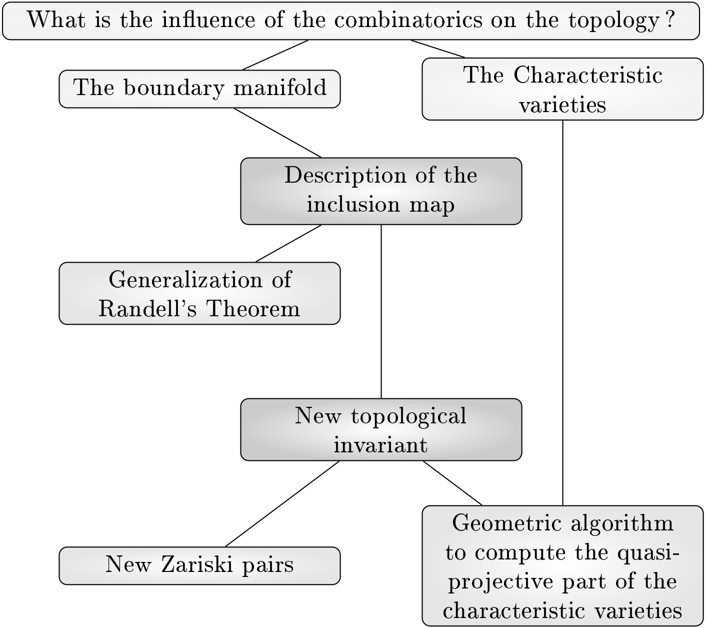
Strategy & plane



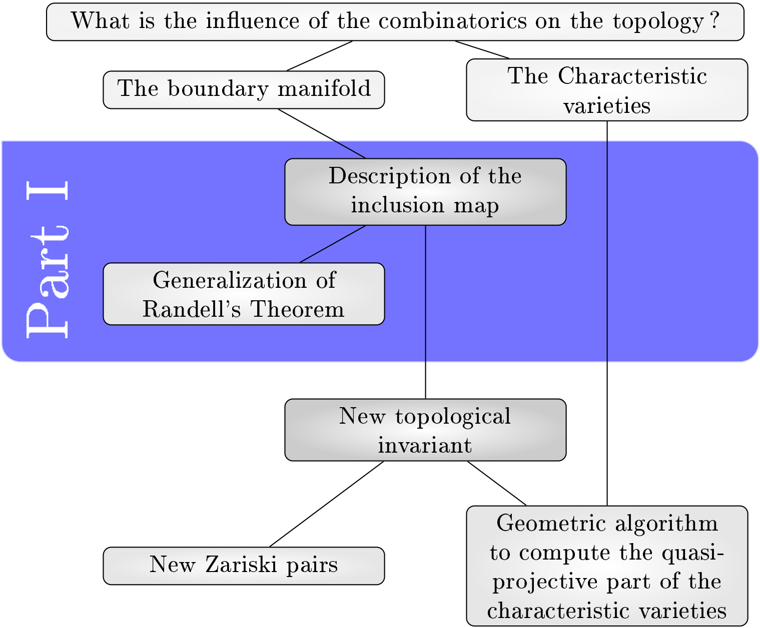
Strategy & plane



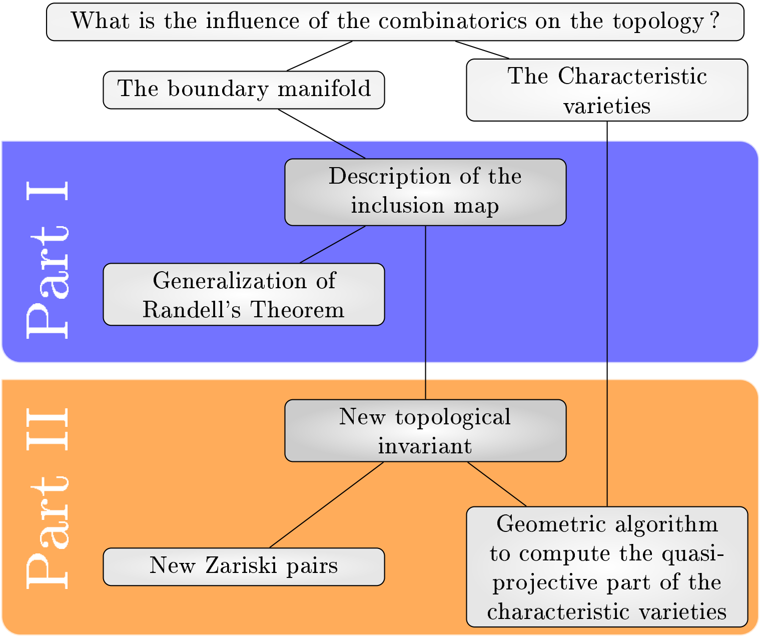
Strategy & plane



Strategy & plane



Strategy & plane



Part I

The Inclusion Map

→ Incidence graph & Wiring diagram

→ The boundary manifold

→ Framed & Geometric cycles

→ Description of the inclusion map

→ Presentation of the fundamental group

Incidence graph

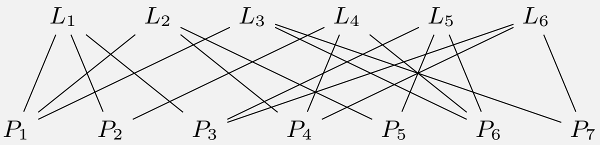
Definition

The incidence graph $\Gamma_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is a non-oriented bipartite graph where the set of vertices decomposes as $V_P(\mathcal{A}) \sqcup V_L(\mathcal{A})$, with :

$$V_P(\mathcal{A}) = \{v_P \mid P \in \mathcal{Q}\}, \text{ and } V_L(\mathcal{A}) = \{v_L \mid L \in \mathcal{A}\}.$$

Example

The incidence graph of Ceva's arrangement is :



$$[\ [1,2,3],[1,4],[1,5,6],[2,4,6],[2,5],[3,4,5],[3,6]\ ]$$

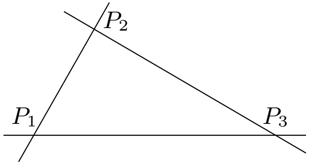
Wiring diagram

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection, and let $\gamma \in \mathbb{C}^2$ be a path containing the $\pi(\mathcal{Q} \cap \mathcal{A}^{\text{aff}})$.

Definition

The *braided wiring diagram* $W_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is :

$$W_{\mathcal{A}} = \pi^{-1}(\gamma) \cap \mathcal{A}^{\text{aff}}.$$



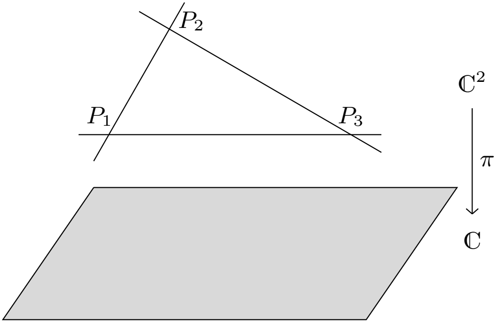
Wiring diagram

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection, and let $\gamma \in \mathbb{C}^2$ be a path containing the $\pi(\mathcal{Q} \cap \mathcal{A}^{\text{aff}})$.

Definition

The *braided wiring diagram* $W_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is :

$$W_{\mathcal{A}} = \pi^{-1}(\gamma) \cap \mathcal{A}^{\text{aff}}.$$



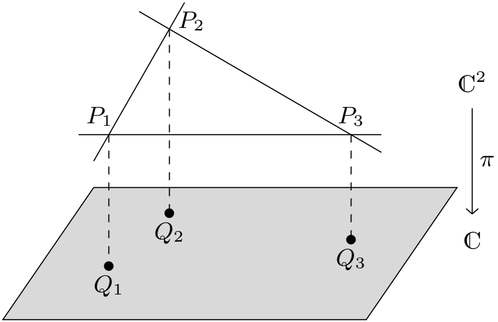
Wiring diagram

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection, and let $\gamma \in \mathbb{C}^2$ be a path containing the $\pi(\mathcal{Q} \cap \mathcal{A}^{\text{aff}})$.

Definition

The *braided wiring diagram* $W_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is :

$$W_{\mathcal{A}} = \pi^{-1}(\gamma) \cap \mathcal{A}^{\text{aff}}.$$



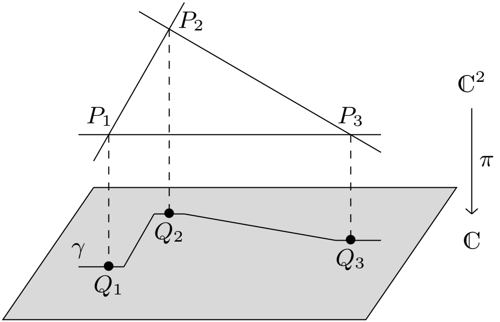
Wiring diagram

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection, and let $\gamma \in \mathbb{C}^2$ be a path containing the $\pi(\mathcal{Q} \cap \mathcal{A}^{\text{aff}})$.

Definition

The *braided wiring diagram* $W_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is :

$$W_{\mathcal{A}} = \pi^{-1}(\gamma) \cap \mathcal{A}^{\text{aff}}.$$



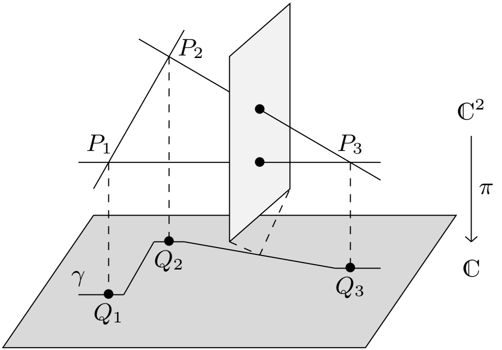
Wiring diagram

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection, and let $\gamma \in \mathbb{C}^2$ be a path containing the $\pi(\mathcal{Q} \cap \mathcal{A}^{\text{aff}})$.

Definition

The *braided wiring diagram* $W_\mathcal{A}$ of an arrangement $\mathcal{A}$ is :

$$W_\mathcal{A} = \pi^{-1}(\gamma) \cap \mathcal{A}^{\text{aff}}.$$



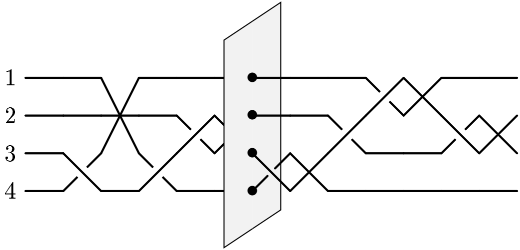
Wiring diagram

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection, and let $\gamma \in \mathbb{C}^2$ be a path containing the $\pi(\mathcal{Q} \cap \mathcal{A}^{\text{aff}})$.

Definition

The *braided wiring diagram* $W_\mathcal{A}$ of an arrangement $\mathcal{A}$ is :

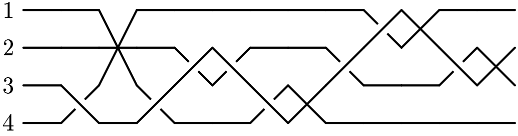
$$W_\mathcal{A} = \pi^{-1}(\gamma) \cap \mathcal{A}^{\text{aff}}.$$

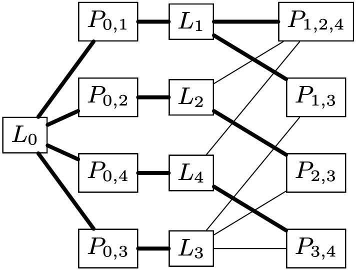


Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\text{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.



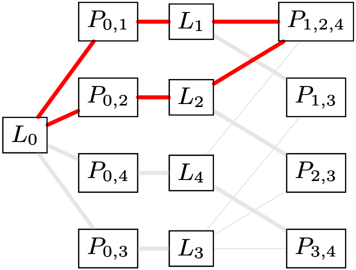


The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\text{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.



The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

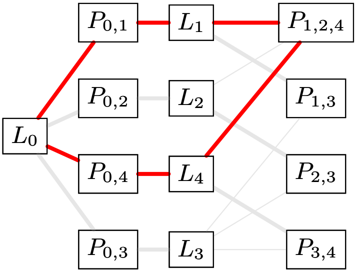
$$\mathscr{E} = \{\xi_{1,2},$$

Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\text{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.

$e_{1,4}$



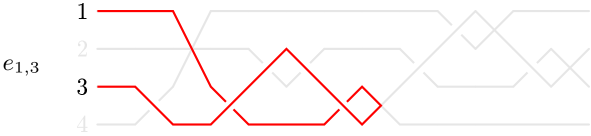
The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

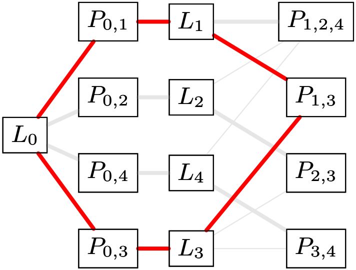
$$\mathscr{E} = \{\xi_{1,2}, \xi_{1,4},$$

Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\mathrm{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.





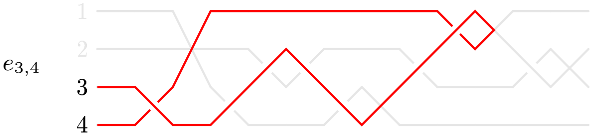
The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

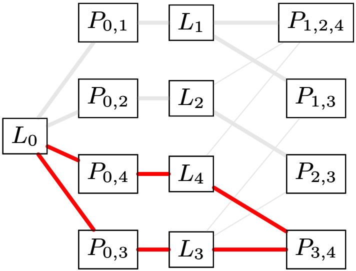
$$\mathscr{E} = \{\xi_{1,2}, \xi_{1,4}, \xi_{1,3},$$

Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\text{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.





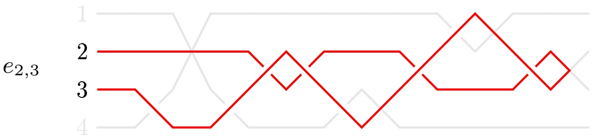
The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

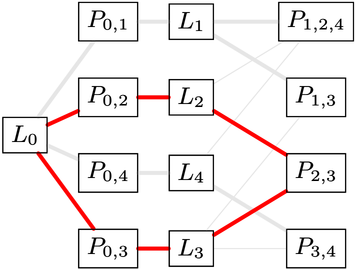
$$\mathscr{E} = \{\xi_{1,2}, \xi_{1,4}, \xi_{1,3}, \xi_{3,4},$$

Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\text{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.





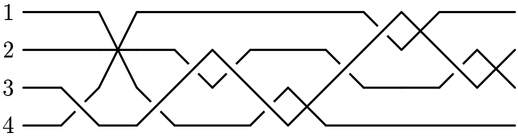
The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

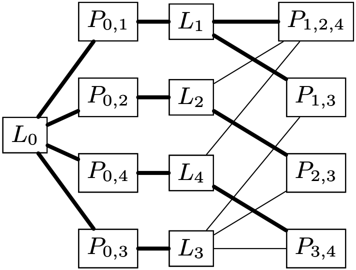
$$\mathscr{E} = \{\xi_{1,2}, \xi_{1,4}, \xi_{1,3}, \xi_{3,4}, \xi_{2,3}\} .$$

Wiring diagram & Incidence graph

Proposition

The graph $\Gamma_{\mathcal{A}^{\text{aff}}}$ is a deformation retract of $W_{\mathcal{A}}$.





The wiring diagram $W_{\mathcal{A}}$ provides a basis of cycles of $\Gamma_{\mathcal{A}}$:

$$\mathscr{E} = \{\xi_{1,2}, \xi_{1,4}, \xi_{1,3}, \xi_{3,4}, \xi_{2,3}\}.$$

Boundary manifold

Definition

The *boundary manifold* $B_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the boundary of a regular neighborhood of $\mathcal{A}$. We have the inclusion :

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}.$$

Proposition

The boundary manifold $B_{\mathcal{A}}$ is a graph manifold –in the sense of F. Waldhausen– over the incidence graph $\Gamma_{\mathcal{A}}$.

Presentation of $\pi_1(B_{\mathcal{A}})$

Let α_i be the meridian associated to L_i, and let $\{\mathfrak{e}_{s,t}\}$ a set of lifts of the $\xi_{s,t} \in \Gamma_{\mathcal{A}}$ in $B_{\mathcal{A}}$.

Theorem

For each singular point $P = P_{i_1,\cdots,i_m}$ with multiplicity m and $i_1 = \min(i_1, \cdots, i_m)$, let

$\mathcal{R}_P = [\alpha_{i_m}^{c_{i_m}}, \cdots, \alpha_{i_2}^{c_{i_2}}, \alpha_{i_1}]$, *where* $c_{i_j} = \mathfrak{e}_{i_1,i_j}$ *for all* $j = 2, \cdots, m$.

The fundamental group of the boundary manifold $B_{\mathcal{A}}$ admits the following presentation :

$$\pi_1(B_{\mathcal{A}}) = \langle \alpha_0, \alpha_1, \cdots, \alpha_n, \mathfrak{e}_{s_1,t_1}, \cdots, \mathfrak{e}_{s_l,t_l} \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}_P \rangle.$$

Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$

Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$

Proposition

The fundamental group of the $B_{\mathcal{A}}$ is generated by :

- the meridians α_i around L_i,
- the cycles $\varepsilon_{s,t}$, indexed by $\mathscr{E}$, coming from $\Gamma_{\mathcal{A}}$.

Where $\varepsilon_{s,t}$ (the *framed cycle*) is in the regular neighborhood of $L_0 \cup L_s \cup L_t$.

Proposition

The fundamental group of the $B_{\mathcal{A}}$ is generated by :

- the meridians α_i around L_i,
- the cycles $\mathcal{E}_{s,t}$, indexed by $\mathscr{E}$, coming from $\Gamma_{\mathcal{A}}$.

Where $\mathcal{E}_{s,t}$ (the *geometric cycle*) is a projection of $e_{s,t}$ in $B_{\mathcal{A}}$.

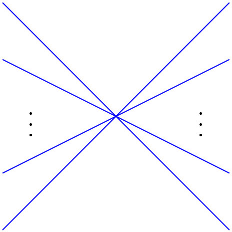
Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$
2. Difference between framed and geometric cycles

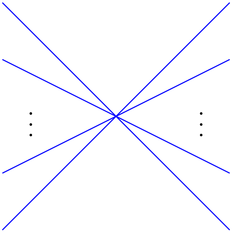
Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$
2. Difference between framed and geometric cycles

Framed cycle

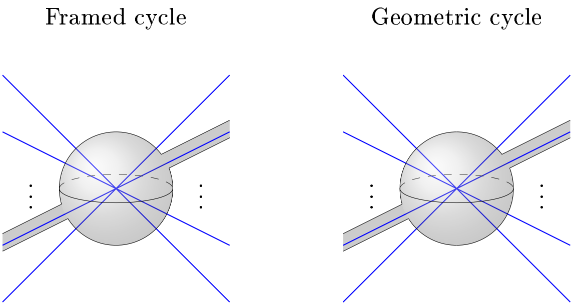


Geometric cycle



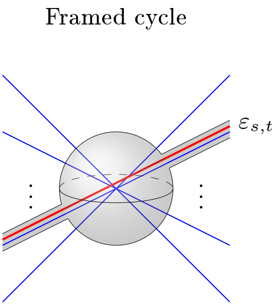
Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$
2. Difference between framed and geometric cycles



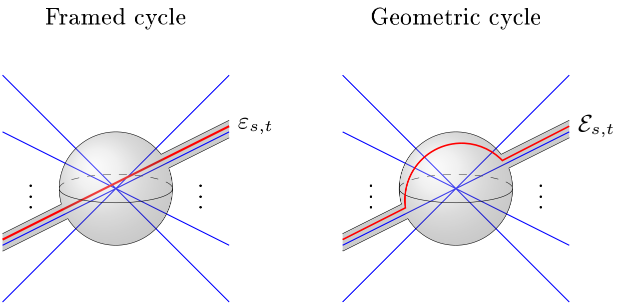
Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$
2. Difference between framed and geometric cycles



Framed & Geometric cycles

1. Generating systems of the boundary manifold $B_{\mathcal{A}}$
2. Difference between framed and geometric cycles



The unknotting map

Definition

Let δ be the unknotting map defined by :

$$\delta : \begin{cases} B_{\mathcal{A}} & \longrightarrow & B_{\mathcal{A}} \\ \alpha_i & \longmapsto & \alpha_i \\ \varepsilon_{s,t} & \longmapsto & \mathcal{E}_{s,t} \end{cases}$$

Proposition

The unknotting map can be described by :

$$\delta(\varepsilon_{s,t}) = \delta^l_{s,t} \cdot \varepsilon_{s,t} \cdot \delta^r_{s,t},$$

where $\delta^l_{s,t}$ and $\delta^r_{s,t}$ are explicit products of the α_i's and the ε's.

The unknotting map

Definition

Let δ be the unknotting map defined by :

$$\delta : \begin{cases} B_{\mathcal{A}} & \longrightarrow & B_{\mathcal{A}} \\ \alpha_i & \longmapsto & \alpha_i \\ \varepsilon_{s,t} & \longmapsto & \mathcal{E}_{s,t} \end{cases}$$

Proposition

The unknotting map can be described by :

$$\delta(\varepsilon_{s,t}) = \delta^l_{s,t} \cdot \varepsilon_{s,t} \cdot \delta^r_{s,t},$$

where $\delta^l_{s,t}$ and $\delta^r_{s,t}$ are explicit products of the α_i's and the ε's.

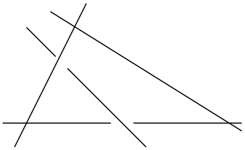
Construction of μ_ε

Definition

For each $\varepsilon_{s,t}$ we define $\mu_{s,t}$ by :

$$\mu_{s,t} = \prod_{\varsigma \in S_{\varepsilon_{s,t}}} a_\varsigma^{e(\varsigma)},$$

where $S_{\varepsilon_{s,t}}$ is the set of the arcs overcrossing $\varepsilon_{s,t}$ in $W_\mathcal{A}$, a_ς is the meridian of the arc ς, and $e(\varsigma)$ the orientation of the crossing.



Proposition

The image of $\mathcal{E}_{s,t}$ by the inclusion map is then $\mu_{s,t}$.

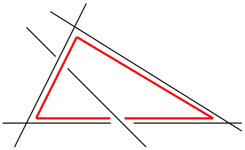
Construction of μ_ε

Definition

For each $\varepsilon_{s,t}$ we define $\mu_{s,t}$ by :

$$\mu_{s,t} = \prod_{\varsigma \in S_{\varepsilon_{s,t}}} a_\varsigma^{e(\varsigma)},$$

where $S_{\varepsilon_{s,t}}$ is the set of the arcs overcrossing $\varepsilon_{s,t}$ in $W_\mathcal{A}$, a_ς is the meridian of the arc ς, and $e(\varsigma)$ the orientation of the crossing.



Proposition

The image of $\mathcal{E}_{s,t}$ by the inclusion map is then $\mu_{s,t}$.

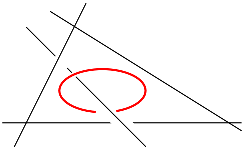
Construction of μ_ε

Definition

For each $\varepsilon_{s,t}$ we define $\mu_{s,t}$ by :

$$\mu_{s,t} = \prod_{\varsigma \in S_{\varepsilon_{s,t}}} a_\varsigma^{e(\varsigma)},$$

where $S_{\varepsilon_{s,t}}$ is the set of the arcs overcrossing $\varepsilon_{s,t}$ in $W_\mathcal{A}$, a_ς is the meridian of the arc ς, and $e(\varsigma)$ the orientation of the crossing.



Proposition

The image of $\mathcal{E}_{s,t}$ by the inclusion map is then $\mu_{s,t}$.

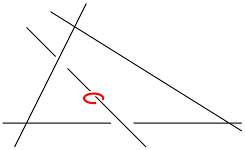
Construction of μ_ε

Definition

For each $\varepsilon_{s,t}$ we define $\mu_{s,t}$ by :

$$\mu_{s,t} = \prod_{\varsigma \in S_{\varepsilon_{s,t}}} a_\varsigma^{e(\varsigma)},$$

where $S_{\varepsilon_{s,t}}$ is the set of the arcs overcrossing $\varepsilon_{s,t}$ in $W_\mathcal{A}$, a_ς is the meridian of the arc ς, and $e(\varsigma)$ the orientation of the crossing.



Proposition

The image of $\mathcal{E}_{s,t}$ by the inclusion map is then $\mu_{s,t}$.

Descriptions of i_*

For $i = 0, \ldots, n$, let α_i be the meridians of the lines and let $\{\varepsilon_{s,t}\}$ (resp. $\{\mathcal{E}_{s,t}\}$) be a set of cycles indexed by a generating system $\mathscr{E}$ of cycles of the incidence graph $\Gamma_{\mathcal{A}}$.

Theorem (Florens, ____, Marco)

The map induced by the inclusion on the fundamental groups can be described by :

$$i_* : \begin{cases} \pi_1(B_{\mathcal{A}}) & \longrightarrow & \pi_1(E_{\mathcal{A}}) \\ \alpha_i & \longmapsto & \alpha_i \\ \varepsilon_{s,t} & \longmapsto & \left(\delta^l_{s,t}\right)^{-1} \cdot \mu_{s,t} \cdot \left(\delta^r_{s,t}\right)^{-1} \end{cases} \qquad \text{FRAMED CYCLES}$$

$$i_* : \begin{cases} \pi_1(B_{\mathcal{A}}) & \longrightarrow & \pi_1(E_{\mathcal{A}}) \\ \alpha_i & \longmapsto & \alpha_i \\ \mathcal{E}_{s,t} & \longmapsto & \mu_{s,t} \end{cases} \qquad \text{GEOMETRIC CYCLES}$$

V. Florens, B. Guerville-Ballé and M. Marco-Buzunàriz, On complex line arrangements and their boundary manifolds

Generalization of Randell's Theorem

For $i = 1, \dots, n$, let α_i be the meridians of the lines L_i. For any singular point $P = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$ with $i_1 = \nu(P)$, let

$$\mathcal{R}'_P = [\alpha_{i_m}^{\mu_{i_1,i_m}}, \cdots, \alpha_{i_2}^{\mu_{i_1,i_2}}, \alpha_{i_1}].$$

Theorem (Florens, ____, Marco)

The fundamental group of $E_\mathcal{A}$ admits the following presentation :

$$\pi_1(E_\mathcal{A}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}'_P \rangle.$$

Corollary (Randell)

Let $\mathcal{A}$ be a complexified real arrangement. The fundamental group of $E_\mathcal{A}$ admits the following presentation :

$$\pi_1(E_\mathcal{A}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P_i \in \mathcal{P}} [\alpha_{i_m}, \cdots, \alpha_{i_2}, \alpha_{i_1}] \rangle,$$

where $P_i = L_{i_1} \cap \cdots \cap L_{i_m}$.

Generalization of Randell's Theorem

For $i = 1, \dots, n$, let α_i be the meridians of the lines L_i. For any singular point $P = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$ with $i_1 = \nu(P)$, let

$$\mathcal{R}'_P = [\alpha_{i_m}^{\mu_{i_1,i_m}}, \cdots, \alpha_{i_2}^{\mu_{i_1,i_2}}, \alpha_{i_1}].$$

Theorem (Florens, _____, Marco)

The fundamental group of $E_\mathcal{A}$ admits the following presentation :

$$\pi_1(E_\mathcal{A}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}'_P \rangle.$$

Corollary (Randell)

Let $\mathcal{A}$ be a complexified real arrangement. The fundamental group of $E_\mathcal{A}$ admits the following presentation :

$$\pi_1(E_\mathcal{A}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P_i \in \mathcal{P}} [\alpha_{i_m}, \cdots, \alpha_{i_2}, \alpha_{i_1}] \rangle,$$

where $P_i = L_{i_1} \cap \cdots \cap L_{i_m}$.

Part II

Topological Invariant

→ The topological invariant

→ Computability of the invariant

→ Zariski pairs

→ Characteristic varieties

Inner-cyclic characters

Let $\hat{\mathcal{A}} \subset \widehat{\mathbb{CP}}^2$ be the blow-up of $\mathcal{A} \subset \mathbb{CP}^2$, and let ξ be a character on the complement of $\mathcal{A}$:

$$\xi : \pi_1(E_{\mathcal{A}}) \longrightarrow \mathbb{C}^*.$$

Definition

A component $L \in \hat{\mathcal{A}}$ is *inner-unramified* for the character ξ if :

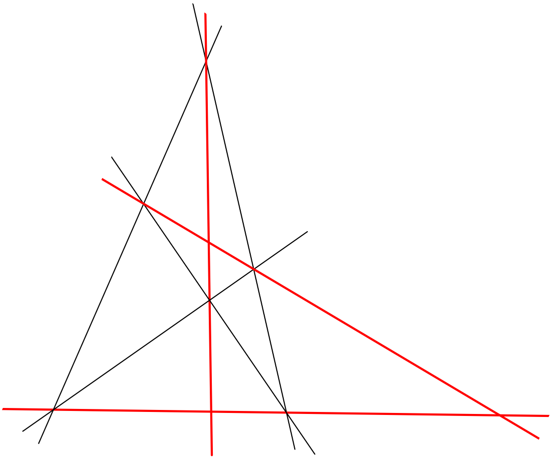
- ξ takes value 1 for its meridian,
- for all $L' \in \hat{\mathcal{A}}$, $(L' \cap L \neq \emptyset) \Rightarrow \xi(\alpha_{L'}) = 1$.

The set of the inner-unramified components of $\hat{\mathcal{A}}$ is $\mathcal{U}_\xi$. Let $\hat{\Gamma}_{\mathcal{U}_\xi}$ be the restriction of $\hat{\Gamma}_{\mathcal{A}}$ to $\mathcal{U}_\xi$.

Definition

A character ξ is *inner-cyclic* if it is torsion and $b_1(\hat{\Gamma}_{\mathcal{U}_\xi}) > 0$.

Inner-cyclic example : Ceva-7



$\xi(\alpha_L) = 1$ (red)

$\xi(\alpha_L) = -1$ (black)

Main result : topological invariant

Definition

A *nearby cycle* $\widetilde{\gamma}$, for character ξ, is a lift of a cycle $\gamma \in \hat{\Gamma}_{\mathcal{U}_\xi}$ contained in a regular neighborhood of $\bigcup_{L \in \mathcal{U}_\xi} L$.

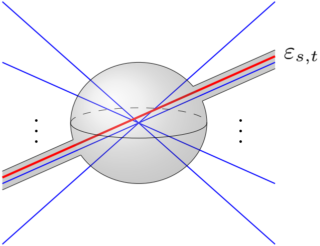
Theorem (Artal, Florens, ____)

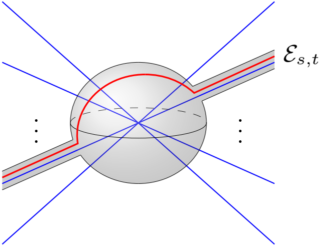
- *ξ an inner-cyclic character,*
- *$\widetilde{\gamma}$ a nearby cycle associated with ξ,*

The image of $\widetilde{\gamma}$ by ξ is a topological invariant of the ordered and oriented pair $(\widehat{\mathbb{CP}}^2, \hat{\mathcal{A}})$.

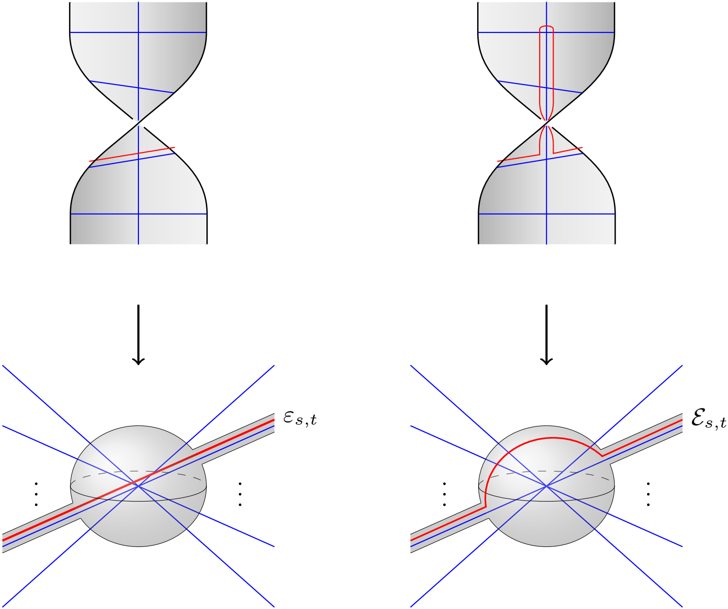
E. Artal Bartolo, V. Florens and B. Guerville-Ballé, A topological invariant of line arrangement

Computability of the invariant





Computability of the invariant



Ordered and oriented NC-Zariski pair I

Let C_4 be the following combinatorics :

$$C_4 = [\ [1,2,3,7],[1,4,8],[1,5,9],[1,6],[2,4],[2,5,8],\\ [2,6,9],[3,4,9],[3,5],[3,6,8],[4,5,6,7],[7,8],[7,9],[8,9]\ ],$$

with the inner-cyclic character :

$$\xi : (\alpha_1, \cdots, \alpha_9) \longmapsto (\zeta, \zeta, \zeta, \zeta^2, \zeta^2, \zeta^2, 1, 1, 1),$$

where ζ is a primitive 3^{rd}-root of unity.

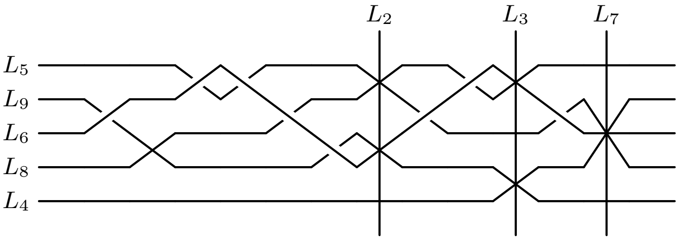
Proposition

The combinatorics C_4 admits two realizations $\mathcal{A}^+$ and $\mathcal{A}^-$ defined by :

$$xyz(x-z)(y-z)(\alpha x-y+z)(\alpha x+\alpha^2 y+z)(x+\alpha^2)(\ x-\alpha^2 y-z) = 0,$$

with $\alpha = \frac{-1\pm i\sqrt{3}}{2}$.

Ordered and oriented NC-Zariski pair II



Proposition

The images of the nearby cycles are :

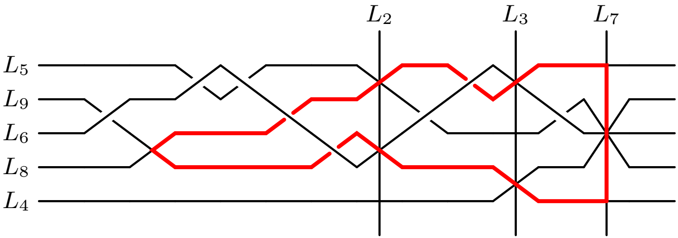
$$\xi(\widetilde{\gamma}_+) = \xi(-\alpha_5 - \alpha_6 + \alpha_6) = \zeta, \qquad \xi(\widetilde{\gamma}_-) = \xi(-\alpha_5 - \alpha_6) = \zeta^2.$$

Theorem (___)

The pair $(\mathcal{A}^+, \mathcal{A}^-)$ is an ordered and oriented NC-Zariski pair.

B. Guerville-Ballé, New Zariski pairs of line arrangements.

Ordered and oriented NC-Zariski pair II



Proposition

The images of the nearby cycles are :

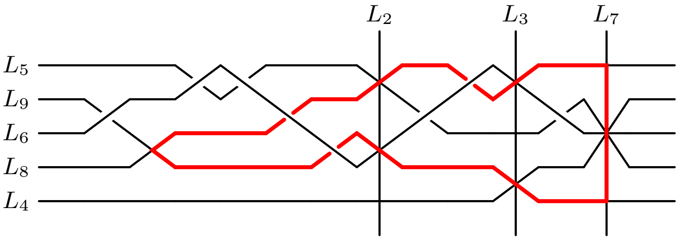
$$\xi(\widetilde{\gamma}_+) = \xi(-\alpha_5 - \alpha_6 + \alpha_6) = \zeta, \qquad \xi(\widetilde{\gamma}_-) = \xi(-\alpha_5 - \alpha_6) = \zeta^2.$$

Theorem (___)

The pair $(\mathcal{A}^+, \mathcal{A}^-)$ is an ordered and oriented NC-Zariski pair.

B. Guerville-Ballé, New Zariski pairs of line arrangements.

Ordered and oriented NC-Zariski pair II



Proposition

The images of the nearby cycles are :

$$\xi(\widetilde{\gamma}_+) = \xi(-\alpha_5 - \alpha_6 + \alpha_6) = \zeta, \qquad \xi(\widetilde{\gamma}_-) = \xi(-\alpha_5 - \alpha_6) = \zeta^2.$$

Theorem (___)

The pair $(\mathcal{A}^+, \mathcal{A}^-)$ *is an ordered and oriented NC-Zariski pair.*

B. Guerville-Ballé, New Zariski pairs of line arrangements.

Ordered NC-Zariski pair

Let C be the following combinatorics :

$$C = [\ [1,2],[1,3,5,7],[1,4,6,8],[1,9],[1,10,11],[2,3,6,9],\\ [2,4,5,10],[2,7,11],[2,8],[3,4],[3,8,11],[3,10],[4,7],[4,9,11],\\ [5,6],[5,8,9],[5,11],[6,7,10],[6,11],[7,8],[7,9],[8,10],[9,10]\ ],$$

with the inner-cyclic character :

$$\xi : (\alpha_1, \cdots, \alpha_{11}) \longmapsto (\zeta, \zeta^4, \zeta^3, \zeta^2, 1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1),$$

where ζ is a primitive 5^{th}-root of unity.

Proposition

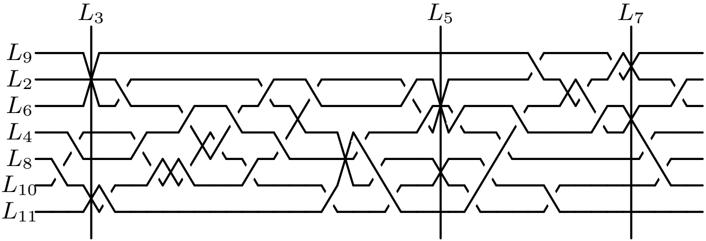
The combinatorics C admits 4 realizations $\mathcal{A}^+$, $\mathcal{A}^-$, $\mathcal{B}^+$ and $\mathcal{B}^-$:

$$xyz(x-z)(y-z)(x+y-z)(-\alpha^3 x+z)(y-\alpha z)((\alpha-1)x-y+z)\\ (-\alpha(\alpha-1)x+y+\alpha(\alpha-1)z)(-\alpha(\alpha-1)x+y-\alpha z)=0.$$

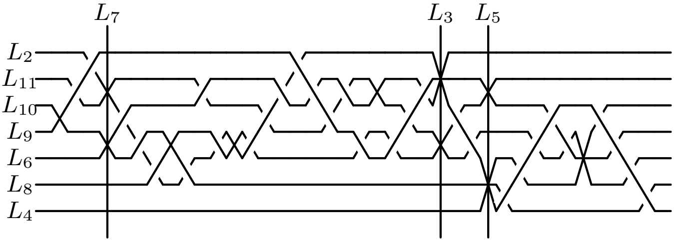
with α a roots of the polynomial $X^4 - X^3 + X^2 - X + 1$.

Wiring diagrams of $\mathcal{A}^+$ and $\mathcal{B}^+$

Braided wiring diagram of $\mathcal{A}^+$:

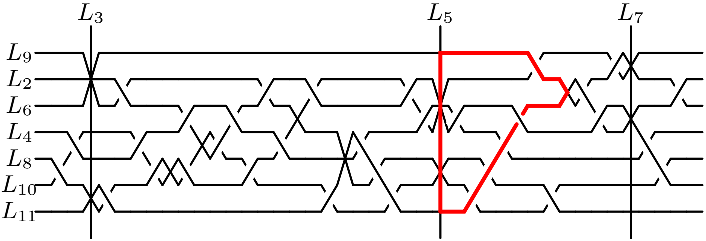


Braided wiring diagram of $\mathcal{B}^+$:

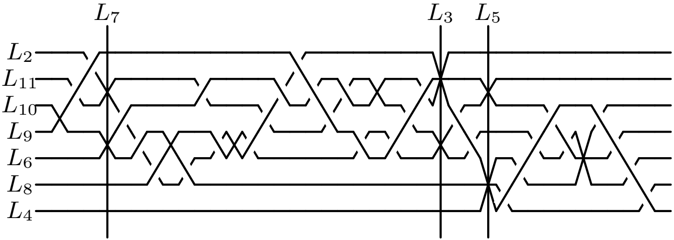


Wiring diagrams of $\mathcal{A}^+$ and $\mathcal{B}^+$

Braided wiring diagram of $\mathcal{A}^+$:

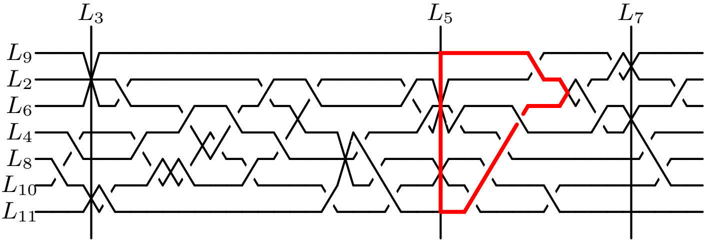


Braided wiring diagram of $\mathcal{B}^+$:

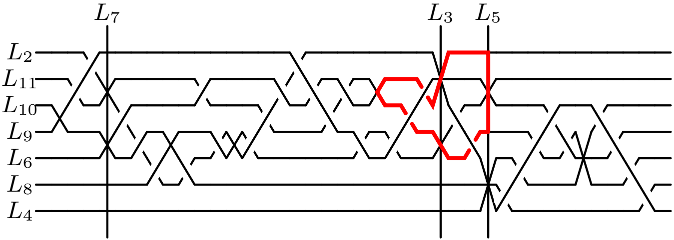


Wiring diagrams of $\mathcal{A}^+$ and $\mathcal{B}^+$

Braided wiring diagram of $\mathcal{A}^+$:



Braided wiring diagram of $\mathcal{B}^+$:



Ordered NC-Zariski pair II

Proposition

The images of the nearby cycles in the complement are :

$$\widetilde{\gamma}_{\mathcal{A}+} = -\alpha_2, \qquad \widetilde{\gamma}_{\mathcal{A}-} = -\alpha_{10} - \alpha_4 - \alpha_8 - \alpha_9,$$
$$\widetilde{\gamma}_{\mathcal{B}+} = -(\alpha_2 + \alpha_9) + \alpha_2, \qquad \widetilde{\gamma}_{\mathcal{B}-} = -(\alpha_2 + \alpha_9).$$

Then, we have :

$$\xi(\widetilde{\gamma}_{\mathcal{A}+}) = \zeta \quad \xi(\widetilde{\gamma}_{\mathcal{A}-}) = \zeta^4 \quad \xi(\widetilde{\gamma}_{\mathcal{B}+}) = \zeta^2 \quad \xi(\widetilde{\gamma}_{\mathcal{B}-}) = \zeta^3$$

Theorem (___)

The pairs $(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$ are ordered NC-Zariski pairs.

Ordered NC-Zariski pair II

Proposition

The images of the nearby cycles in the complement are :

$$\widetilde{\gamma}_{\mathcal{A}+} = -\alpha_2, \qquad \widetilde{\gamma}_{\mathcal{A}-} = -\alpha_{10} - \alpha_4 - \alpha_8 - \alpha_9,$$

$$\widetilde{\gamma}_{\mathcal{B}+} = -(\alpha_2 + \alpha_9) + \alpha_2, \qquad \widetilde{\gamma}_{\mathcal{B}-} = -(\alpha_2 + \alpha_9).$$

Then, we have :

$$\xi(\widetilde{\gamma}_{\mathcal{A}+}) = \zeta \quad \xi(\widetilde{\gamma}_{\mathcal{A}-}) = \zeta^4 \quad \xi(\widetilde{\gamma}_{\mathcal{B}+}) = \zeta^2 \quad \xi(\widetilde{\gamma}_{\mathcal{B}-}) = \zeta^3$$

Theorem (___)

The pairs $(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$ are ordered NC-Zariski pairs.

NC-Zariski pair

Definition

Adding a generic line L_{12} to C passing through $L_1 \cap L_3 \cap L_5 \cap L_7$, we obtain four arrangements denoted by :

$$\mathscr{A}^+, \quad \mathscr{A}^-, \quad \mathscr{B}^+, \quad \mathscr{B}^-.$$

Theorem ( )

The pairs $(\mathscr{A}^{\pm}, \mathscr{B}^{\pm})$ are NC-Zariski pairs.

Theorem ( )

The 4-tuplet $(\mathscr{A}^+, \mathscr{B}^+, \mathscr{A}^-, \mathscr{B}^-)$ is an oriented NC-Zariski 4-tuplet.

NC-Zariski pair

Definition

Adding a generic line L_{12} to C passing through $L_1 \cap L_3 \cap L_5 \cap L_7$, we obtain four arrangements denoted by :

$$\mathscr{A}^+, \quad \mathscr{A}^-, \quad \mathscr{B}^+, \quad \mathscr{B}^-.$$

Theorem (___)

The pairs $(\mathscr{A}^\pm, \mathscr{B}^\pm)$ are NC-Zariski pairs.

Theorem (___)

The 4-tuplet $(\mathscr{A}^+, \mathscr{B}^+, \mathscr{A}^-, \mathscr{B}^-)$ is an oriented NC-Zariski 4-tuplet.

NC-Zariski pair

Definition

Adding a generic line L_{12} to C passing through $L_1 \cap L_3 \cap L_5 \cap L_7$, we obtain four arrangements denoted by :

$$\mathscr{A}^+, \quad \mathscr{A}^-, \quad \mathscr{B}^+, \quad \mathscr{B}^-.$$

Theorem (___)

The pairs $(\mathscr{A}^\pm, \mathscr{B}^\pm)$ are NC-Zariski pairs.

Theorem (___)

The 4-tuplet $(\mathscr{A}^+, \mathscr{B}^+, \mathscr{A}^-, \mathscr{B}^-)$ is an oriented NC-Zariski 4-tuplet.

Depth of characters : E. Artal's results

Let $E_{\mathcal{A}}^{\xi}$ (resp. $(\widehat{\mathbb{CP}}^2)^{\xi}$) be the smooth, (resp. smooth, branched), cyclic cover of $E_{\mathcal{A}}$ (resp. $\widehat{\mathbb{CP}}^2$) associated with ξ. Let $\check{\xi}$ be a generator of the group of deck automorphism, and let $\check{\xi}^*$ be the induced map on the cohomology groups.

Definition

The quasi-projective depth of a character ξ on $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$ is defined by :

$$\overline{\mathrm{depth}}(\xi) = \dim \operatorname{coker} \left(\mathrm{H}^1\left((\widehat{\mathbb{CP}}^2)^{\xi};\mathbb{C}\right)^{\check{\xi}_*} \to \mathrm{H}^1\left(E_{\mathcal{A}}^{\xi};\mathbb{C}\right)^{\check{\xi}_*} \right).$$

Theorem (Artal)

Let ξ be a character on $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$, then :

$$\overline{\mathrm{depth}}(\xi) = \operatorname{corank} K_{\mathcal{A}}^{\check{\xi}^*},$$

where $K_{\mathcal{A}}^{\check{\xi}^*}$ is a square matrix determined by the combinatorics and the images by ξ of a basis of the nearby cycles of $\mathcal{A}$.

Computation for Ceva-7

Let ξ be the character on the complement of the Ceva-7 arrangement previously presented :

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (1, -1, -1, 1, -1, -1, 1).$$

The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$:

$$K_{\mathcal{A}}^{\xi} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \xi(\varepsilon_{3,6}) \\ 1 & \xi(\varepsilon_{3,6})^{-1} & -1 \end{pmatrix}$$

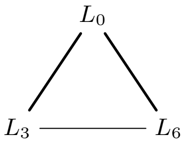
$$\overline{\text{depth}}(\xi) = \operatorname{corank} K_{\mathcal{A}}^{\xi} = 2.$$

Computation for Ceva-7

Let ξ be the character on the complement of the Ceva-7 arrangement previously presented :

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (1, -1, -1, 1, -1, -1, 1).$$

The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$:



$$K^\xi_{\mathcal{A}} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \xi(\varepsilon_{3,6}) \\ 1 & \xi(\varepsilon_{3,6})^{-1} & -1 \end{pmatrix}$$

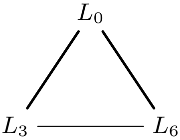
$$\overline{\text{depth}}(\xi) = \operatorname{corank} K^\xi_{\mathcal{A}} = 2.$$

Computation for Ceva-7

Let ξ be the character on the complement of the Ceva-7 arrangement previously presented :

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (1, -1, -1, 1, -1, -1, 1).$$

The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$:



$$K_{\mathcal{A}}^{\xi} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \xi(\varepsilon_{3,6}) \\ 1 & \xi(\varepsilon_{3,6})^{-1} & -1 \end{pmatrix}$$

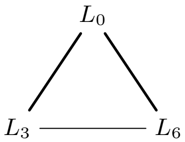
$$\overline{\mathrm{depth}}(\xi) = \operatorname{corank} K_{\mathcal{A}}^{\xi} = 2.$$

Computation for Ceva-7

Let ξ be the character on the complement of the Ceva-7 arrangement previously presented :

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (1, -1, -1, 1, -1, -1, 1).$$

The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$:



$$K^{\xi}_{\mathcal{A}} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \xi(\varepsilon_{3,6}) \\ 1 & \xi(\varepsilon_{3,6})^{-1} & -1 \end{pmatrix}$$

$$\overline{\text{depth}}(\xi) = \operatorname{corank} K^{\xi}_{\mathcal{A}} = 2.$$

Conclusion

• Two explicit descriptions of the inclusion map, and a generalization of Randell's presentation of the fundamental group of $E_{\mathcal{A}}$

Conclusion

- Two explicit descriptions of the inclusion map, and a generalization of Randell's presentation of the fundamental group of $E_{\mathcal{A}}$

- A new topological invariant

For $\widetilde{\gamma}$ a nearby cycle associated with an inner-cyclic character ξ :

$$\xi(\widetilde{\gamma}) \text{ is a topological invariant}$$

Conclusion

• Two explicit descriptions of the inclusion map, and a generalization of Randell's presentation of the fundamental group of $E_{\mathcal{A}}$

• A new topological invariant

• New examples of NC-Zariski pairs

The pairs $(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$ form ordered NC-Zariski pairs

The pairs $(\mathscr{A}^{\pm}, \mathscr{B}^{\pm})$ form NC-Zariski pairs

The arrangement $\mathscr{A}^{+}, \mathscr{A}^{-} \mathscr{B}^{+}$ and $\mathscr{B}^{-}$ form an oriented NC-Zariski 4-tuplet

Conclusion

• Two explicit descriptions of the inclusion map, and a generalization of Randell's presentation of the fundamental group of $E_{\mathcal{A}}$

• A new topological invariant

• New examples of NC-Zariski pairs

• A geometric method to compute the quasi-projective depth of any character

Continuations

- Optimize the programs (`Sage` and `Maple`)

Modify the construction of the wiring diagram

Reduce the set of realizations

Develop a better algorithm to detect the prime combinatorics

Continuations

- Optimize the programs (`Sage` and `Maple`)

- Extend the results to the case of rational algebraic plane curves

Description of the inclusion map of the boundary manifold

Generalization of the topological invariant

Continuations

- Optimize the programs (`Sage` and `Maple`)

- Extend the results to the case of rational algebraic plane curves

- Continue the study of the depth of characters

> Obtain new combinatorial conditions to admit a character with a non null quasi-projective depth

Continuations

- Optimize the programs (`Sage` and `Maple`)

- Extend the results to the case of rational algebraic plane curves

- Continue the study of the depth of characters

- Description of the inclusion map on the twisted homology

Continuations

- Optimize the programs (`Sage` and `Maple`)

- Extend the results to the case of rational algebraic plane curves

- Continue the study of the depth of characters

- Description of the inclusion map on the twisted homology

- Continue the study of the combinatorics and the characteristic varieties

End

Thank you for your attention.