On the boundary manifold of a complex line arrangement

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Plan

Definitions

- Complex line arrangement $\mathcal{A}$: union of line in $\{L_0, \cdots, L_n\}$ in $\mathbb{CP}^2$.

- Complexified real arrangement : L_i have real equations.

- $E(\mathcal{A}) = \left(\mathbb{CP}^2 - \mathring{T}(\mathcal{A})\right)$: exterior of $\mathcal{A}$.

- *Topology* of $\mathcal{A}$: topological type of the exterior of the arrangement.

It is determined by the pair $(\mathbb{CP}^2, \mathcal{A})$.

Question

What determines the topology of an arrangement ?

Fundamental groups vs Combinatorics

- Fundamental group of $E(\mathcal{A})$: topological invariant.

Too complicated.

- The *combinatorics* of an arrangement is the description of the multiple points and the incidence relations.

Question

Does the combinatorics determines the fundamental group ?

No. Rybnikov (1998).

Main result

Let $\mathcal{A}$ a line arrangement, we define the *boundary manifold* of $\mathcal{A}$ by : $M(\mathcal{A}) = \partial E(\mathcal{A})$. It can be constructed only from the combinatorics of $\mathcal{A}$.

Theorem

The map induced by i on the fundamental group,

$$i_* : \pi_1(M(\mathcal{A})) \longrightarrow \pi_1(E(A)),$$

can be explicitly described from a specific presentation of $\pi_1(M(\mathcal{A}))$.

Didactic example

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$$L_0 = \{z = 0\}\,, \qquad L_1 = \{-(i+2)x + (2i+3)y = 0\}\,,$$
$$L_2 = \{-x + (i+2)y = 0\}\,, \quad L_3 = \{-x + 3y + iz = 0\}$$
$$L_4 = \{-x + (2i+2)y = 0\}\,.$$

Incidence graph

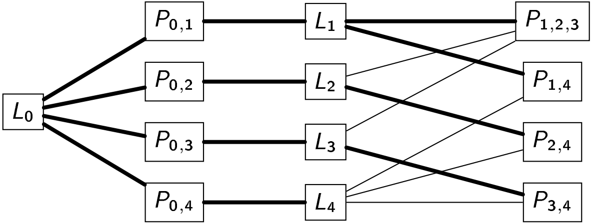
- Incidence graph $\Gamma_{\mathcal{A}}$: graph containing the combinatorics of an arrangement.

- $\Gamma_{\mathcal{A}}$: non-oriented, bipartite, and vertices $V(\mathcal{A}) = V_P(\mathcal{A}) \coprod V_L(\mathcal{A})$.

$$V_P(\mathcal{A}) = \{v_P \mid P \in \overline{\mathcal{P}}\}, \quad V_L(\mathcal{A}) = \{v_L \mid L \in \mathcal{A}\},$$

where : - $\overline{\mathcal{P}}$: set of the singular points of $\mathcal{A}$.

- Edges of $\Gamma(\mathcal{A})$: $Y(L,P)$, with $L \in \mathcal{A}_P = \{\ L_i \ \mid\ P \in L_i\ \}$.



Construction of the boundary manifold

The boundary manifold can be constructed as a graph manifold over $\Gamma_{\mathcal{A}}$.

- For point-vertex v_P, we take a 3-sphere without an Hopf link with multiplicity of P components. With fundamental group :

$$< y_{k_1}, \cdots, y_{k_m} \mid [y_{k_1}, \cdots, y_{k_m}] >$$

- For line-vertex v_L, we take a S^1-bundle over the line L without the singular point of $\mathcal{A}$. With fundamental group :

$$< x_{k_1}, \cdots, x_{k_l}, \alpha_k \mid \forall i \in \{k_1, \cdots, k_l\},\ \alpha_k^{-1} x_i \alpha_k = x_i >$$

Fundamental group of the boundary manifold

Presentation of the boundary manifold

Let α_i be the meridians around the L_i, and let the $\varepsilon_{(s,t)}$ be in bijection with a cycles basis of $\Gamma_{\mathcal{A}}$; then $\pi_1(M(\mathcal{A}))$ admits the following presentation :

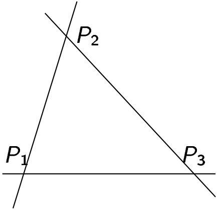
$$\pi_1(M(\mathcal{A})) = < \alpha_1, \cdots, \alpha_n, \varepsilon_{(s_1,t_1)}, \cdots, \varepsilon_{(s_l,t_l)} \mid \forall i \in \{1, \cdots, k\}, \ \mathcal{R}_i >,$$

where - $\mathcal{R}_i = [\alpha_{i_1}, \alpha_{i_2}^{c_{i_2}}, \cdots, \alpha_{i_m}^{c_{i_m}}]$ if $P_i = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$;

- $c_{i_j} = \varepsilon_{(i_1,i_j)}$ for all $j \in \{2, \cdots, m\}$.

Some definitions

- $\{P_1, \cdots, P_k\}$: singular points of $\mathcal{A}$,



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- $\{P_1, \cdots, P_k\}$: singular points of $\mathcal{A}$,

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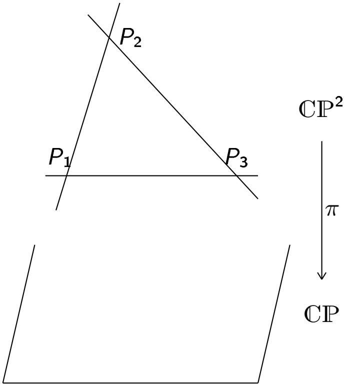

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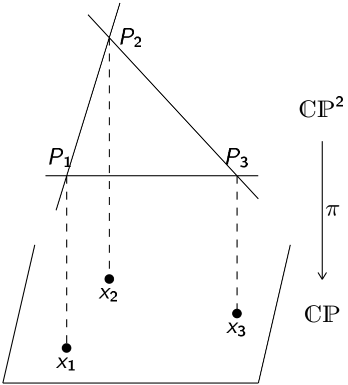

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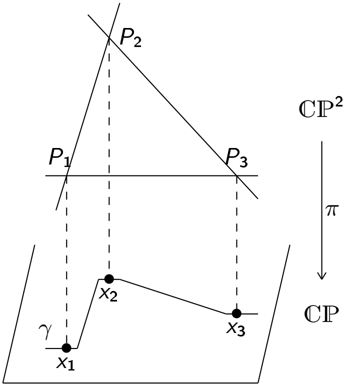

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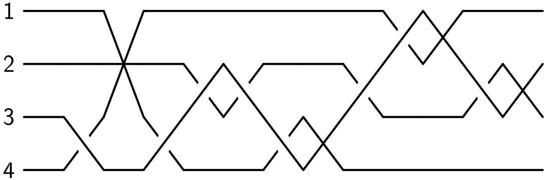
- γ a path of $\mathbb{CP}$ such that : $\forall i,\ \exists t \in [0,1], \gamma(t) = x_i$.

Braided wiring diagram

Braided wiring diagram

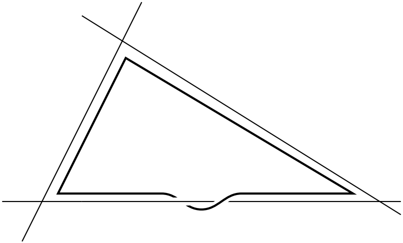
The braided wiring diagram $W_{\mathcal{A}}$ is defined by :

$$W_{\mathcal{A}} = \mathcal{A} \cap \pi^{-1}(\gamma),$$



Unknotted cycles

To each cycle $\varepsilon_{(s,t)}$ of the presentation of $\pi_1(M(\mathcal{A}))$, we associate a word $\sigma(\varepsilon_{(s,t)}) = \delta^l_{(s,t)}\varepsilon_{(s,t)}\delta^r_{\varepsilon_{(s,t)}}$.



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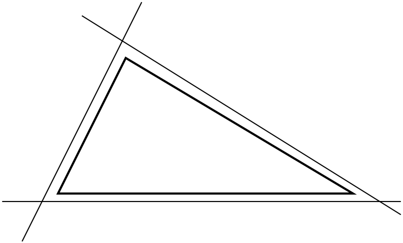
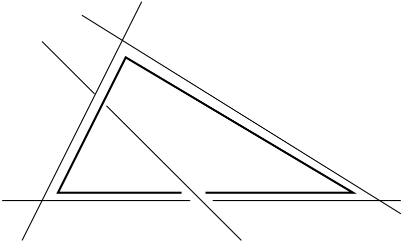


Over arcs and μ_ε

To each cycle ε of the wiring diagram, we associate a word μ_ε.



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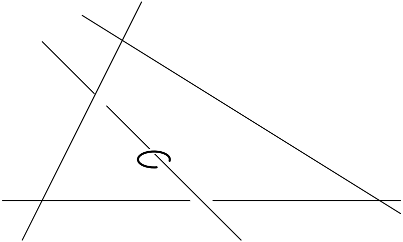


Main results

Theorem

The map $i_* : \pi_1(M(\mathcal{A})) \to \pi_1(E(\mathcal{A}))$ induced by the inclusion is described by :

$$i_* : \begin{cases} \alpha_j & \longmapsto & \alpha_j, \\ \varepsilon_{(s,t)} & \longmapsto & (\delta^l_{(s,t)})^{-1} \mu_{\varepsilon_{(s,t)}} (\delta^r_{\varepsilon_{(s,t)}})^{-1}. \end{cases}$$

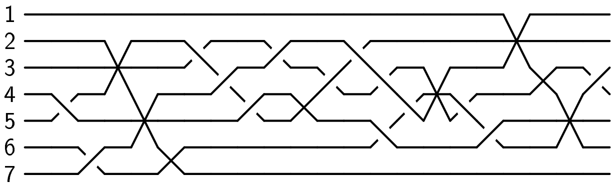
where α_j and $\varepsilon_{(s,t)}$ are the generators of the presentation of $\pi_1(M(\mathcal{A}))$.

Property

Moreover i_* is onto, with kernel generated by $\delta^l_{\varepsilon_{(s,t)}} \varepsilon_{(s,t)} \delta^r_{\varepsilon_{(s,t)}} \mu^{-1}_{\varepsilon_{(s,t)}}$.

Computation I

The braided wiring diagram :

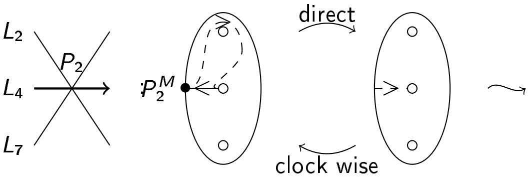


Computation I

The braided wiring diagram :

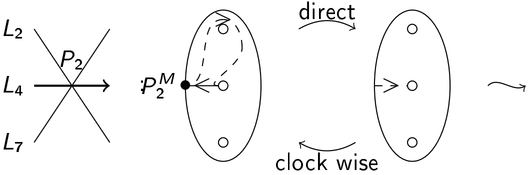
The example of the cycle $\varepsilon_{(4,5)}$.

Computation II

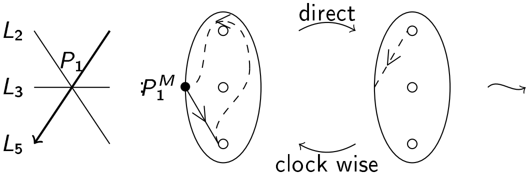


$$\delta^I_{(4,5)} = \varepsilon_{2,4}^{-1} x_2^{-1} \varepsilon_{2,4}$$

Computation II

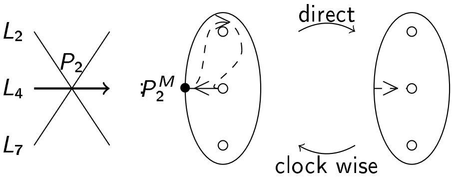


$$\delta^{l}_{(4,5)} = \varepsilon_{2,4}^{-1} x_2^{-1} \varepsilon_{2,4}$$

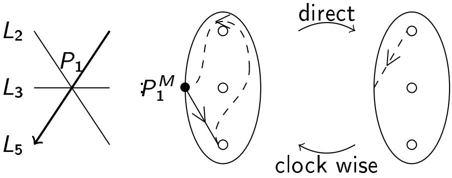


$$\delta^{r}_{(4,5)} = \varepsilon_{2,5}^{-1} \varepsilon_{2,3} x_3 \varepsilon_{2,3}^{-1} x_2 \varepsilon_{2,5}$$

Computation II



$\sim$ $\delta^l_{(4,5)} = \varepsilon_{2,4}^{-1} x_2^{-1} \varepsilon_{2,4}$



$\sim$ $\delta^r_{(4,5)} = \varepsilon_{2,5}^{-1} \varepsilon_{2,3} x_3 \varepsilon_{2,3}^{-1} x_2 \varepsilon_{2,5}$

$$\boxed{\mathcal{R}_{4,5} \ : \ \varepsilon_{4,5} = x_2 x_7^{-1} x_4 x_7^2 x_4^{-1} x_2^{-1} x_3^{-1}.}$$

Final result

Presentation

$$\pi_1(E(\mathcal{A}^+)) =< x_1, \cdots, x_7,$$

$$\varepsilon_{2,3}, \varepsilon_{2,5}, \varepsilon_{2,4}, \varepsilon_{2,7}, \varepsilon_{2,6}, \varepsilon_{4,5}, \varepsilon_{3,6}, \varepsilon_{3,7}, \varepsilon_{1,5}, \varepsilon_{1,7}, \varepsilon_{1,3}, \varepsilon_{1,4}, \varepsilon_{1,6} \mid$$

$$\mathcal{R}_{2,3}, \mathcal{R}_{2,5}, \mathcal{R}_{2,4}, \mathcal{R}_{2,7}, \mathcal{R}_{2,6}, \mathcal{R}_{4,5}, \mathcal{R}_{3,6}, \mathcal{R}_{3,7}, \mathcal{R}_{1,5}, \mathcal{R}_{1,7}, \mathcal{R}_{1,3}, \mathcal{R}_{1,4}, \mathcal{R}_{1,6},$$

$$[x_2, x_3^{\varepsilon_{2,3}}, x_5^{\varepsilon_{2,5}}], [x_2, x_4^{\varepsilon_{2,4}}, x_7^{\varepsilon_{2,7}}], [x_2, x_6^{\varepsilon_{2,6}}], [x_4, x_5^{\varepsilon_{4,5}}],$$

$$[x_3, x_5^{\varepsilon_{3,5}}, x_7^{\varepsilon_{3,7}}], [x_1, x_5^{\varepsilon_{1,5}}, x_7^{\varepsilon_{1,7}}], [x_1, x_3^{\varepsilon_{1,3}}], [x_1, x_4^{\varepsilon_{1,4}}, x_6^{\varepsilon_{1,6}}] >$$

The End

Thank you for your attention